Degree mixing and the enhancement of synchronization in complex weighted networks

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Real networks often consist of local units interacting with each other by means of heterogeneous connections. In many cases, furthermore, such networks feature degree mixing properties, i.e., the tendency of nodes with high degree (with low degree) to connect with connectivity peers (with highly connected nodes). Such degree-degree correlations may have an important influence in the spreading of information or infectious agents on a network. We explore the role played by these correlations for the synchronization of networks of coupled dynamical systems. Using a stochastic optimization technique, we find that the value of degree mixing providing optimal conditions for synchronization depends on the weighted coupling scheme. We also show that a minimization of the assortative coefficient may induce a strong destabilization of the synchronous state. We illustrate our findings for weighted networks with scale free and random topologies.

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In recent years, complex networks have provided a challenging framework for the study of collective (synchronized) behaviors, based on the interplay between complexity in the overall topology and local dynamical properties of the coupled units [1]. In particular, complex wirings have been proven to enhance (with respect to regular topologies) the ability of a network to synchronize [2,3]. Initially, this enhancement was attributed to the decrease in the average network distance between nodes. In fact, synchronization is affected by many topological features such as the degree distribution, characteristic path length, betweenness centrality, and weight distributions among others [4–6].

In many real-world networks, vertices also exhibit a tendency to be connected to other vertices with similar (or dissimilar) degree, i.e., nodes with many connections tend to connect to nodes with many (or few) connections [7]. Social networks for instance, tend to exhibit positive degree-degree correlations—often called assortative mixing—whereas biological and technological networks display a disassortative mixing. It has been found that the degree mixing strongly determines the extent to which information or an infectious disease is contained within a core group, or spread to the rest of the population [7,8].

In this paper we explore the role of degree-degree correlations for the synchronization of weighted complex networks. Namely, (i) we assess the network propensity for synchronization as a function of its assortative degree and (ii) we give evidence that the degree mixing providing optimal conditions for synchronization depends on the weighted coupling scheme. We also show that, (iii) minimizing the assortative behavior of a network may induce a strong destabilization of the synchronous state.

We start by considering a generic network of N coupled maps, whose dynamical evolution is ruled by

$$\mathbf{x}_{t+1}^{i}=\mathbf{f}(\mathbf{x}_{t}^{i})-\sigma\sum_{j=1}^{N}G_{ij}\mathbf{H}(\mathbf{x}_{t}^{j}),\quad i=1,\ldots,N, \tag{1}$$

where $\mathbf{x}_{t+1}^{j}=\mathbf{f}(\mathbf{x}_{t}^{i})$ governs the local dynamics of each map, the function $\mathbf{H}[\mathbf{x}]$ defines the nature of coupling between nodes, σ is the coupling strength, and G is the coupling matrix accounting for the network's topology. G is assumed to be zero row sum [which ensures the existence of an invariant synchronization manifold wherein the coupling term in Eq. (1) exactly vanishes], and to have a real spectrum of eigenvalues.

If diagonal elements $G_{ii}=-\Sigma_{j\neq i}G_{ij}\geqslant 0$, G has a spectrum of real and non-negative eigenvalues, with the smallest eigenvalue $\lambda_1=0$ because $\Sigma_j G_{ij}=0\,\forall\, i$, and $\lambda_2>0$ if the network is connected. The eigenvalues can be then ordered as $0=\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_N$ [9]. According to the criteria of the master stability function [10], the propensity of a network to synchronize can be inspected by the eigenratio λ_N/λ_2 of the coupling matrix G (originally assumed as diagonalizable but not necessarily symmetric): the smaller the eigenratio is, the higher the chance of having a stable synchronization for some σ [2,10]. Recently, Ref. [11] has extended the above formalism also to the case of nondiagonalizable coupling matrices.

Traditionally, the oscillators are coupled with uniform and undirected coupling strengths (unweighted links). There are, however, paradigmatic cases where a weighting or an asymmetry in the connections has relevant consequences in determining the network's dynamics. In ecological systems, for instance, the heterogeneity of weights and in prey-predator interactions plays a crucial role in determining the food web stability [12]. Similarly, the natural differences of neurons and their synaptic weights connections play an important role in the capabilities of transmission and information processing in neural networks [13].

Recent studies have revealed the strong influence of weighted and asymmetric coupling configurations on the emergence of coherent global behavior [5,6]. Motivated by empirical observations of metabolic and airport networks [14,15], which have revealed that the average link weight between nodes i and j scales with powers of the product of the corresponding degrees [$\langle W_{ij}\rangle\sim(k_ik_j)^{\phi}$, with ϕ depending on the observed network], we consider here a weighted coupling term given by [16]

$$\sum_{j=1}^{N} G_{ij}\mathbf{H}[\mathbf{x}_j] = \frac{1}{\mathcal{K}_i}\sum_{j\in\mathcal{N}_i} W_{ij}\mathbf{H}[\mathbf{x}_j], \tag{2}$$

where $W_{ij}=(k_i k_j)^{\theta}$, and $\mathcal{N}_i$ is the set of neighbors to the ith node. θ is a real tunable parameter that controls the heterogeneity of the input strength of nodes. It has to be noticed that an optimal condition $\theta=-1$ for synchronization was recently found [16] for $\mathcal{K}_i=\mathcal{K}=1$ on uncorrelated SF networks.

Under many circumstances it is convenient to normalize the diagonal elements of the coupling matrix to $G_{ii}=1$. Such normalization prevents the coupling term from being arbitrarily large (or arbitrarily small) for all possible network topologies, thus avoiding the local influence of the environment on the dynamics to scale with the number of connections. Such a normalized coupling can be obtained by setting $\mathcal{K}_i=\sum_{j\in\mathcal{N}_i} W_{ij}$.

Although the coupling matrix G becomes asymmetric, it can be written as a product $G=DL$, where L is a zero row-sum matrix with off-diagonal entries $L_{ij}=-W_{ij}$, and $D=\mathrm{diag}\{\frac{1}{\sum_j W 1j},\ldots,\frac{1}{\sum_j W_{Nj}}\}$. The eigenvalue spectrum of G is the same as that obtained from the matrix $W=D^{1/2}LD^{1/2}$, and therefore is real with nonnegative values, and all the arguments of the master stability function approach apply. In this case, the eigenvalues also verify $\lambda_i \leqslant 2\,\forall i$ [9], which warrants that the largest eigenvalue λ_N will never diverge, independently of the network size N and for all possible network topologies.

A first attempt at assessing the role of assortative mixing on synchronization was recently provided in Ref. [17] for unweighted and weighted coupling configurations (with a weighting scheme based on powers of the node degrees as in Ref. [5]). In both cases, networks were found to synchronize better when their topology displayed a disassortative mixing, i.e., when nodes with low degree were more likely to be connected to nodes with a high degree. In this paper we complement the study of Ref. [17], and show that the propensity for synchronization in correlated networks depends on the weighting procedure. While we will recover the essential features of Ref. [17], we will show that for a more general weighting scheme, tuning the assortative mixing of a network towards large negative values may induce a strong destabilization of the synchronous state.

Following Refs. [7] the assortative coefficient of a network can be estimated by $r_{\mathrm{net}}=[\sum_{j,k} jk(e_{jk}-q_j q_k)]/\sigma_q^2$, where e_{jk} is the fraction of edges that connect vertices of degrees j and k, the distribution $q_k=\sum_j e_{jk}$ with standard deviation σ_q. For practical purposes of evaluating r on an observed network, this expression can be written in the form [7]

$$r_{\mathrm{net}} = \frac{M^{-1}\sum_i^M j_i k_i - \left[M^{-1}\sum_i^M \frac{1}{2}(j_i+k_i)\right]^2}{M^{-1}\sum_i^M \frac{1}{2}(j_i+k_i) - \left[M^{-1}\sum_i^M \frac{1}{2}(j_i+k_i)\right]^2}, \tag{3}$$

where j_i and k_i are the degrees of the nodes at the ends of the ith edge and M is the total number of edges.

To assess the effect of the assortative mixing on the synchronization of a network, we consider here a randomized ensemble of networks in which the degree distribution $p(k)$ is kept unchanged, while the assortative coefficient is tuned by using a simulated annealing (SA) algorithm [18]. This stochastic optimization technique allows performing an exhaustive search and finding a low-cost configuration without getting trapped in high-cost local minima. To create such an ensemble of networks we successively randomize a network with the algorithm described in Ref. [19]. According to this algorithm, at some finite "temperature" T, we randomly select pair of edges (that do not share a common node) and the attaching nodes are exchanged with probability $\min\{1,\exp(-\Delta E/T)\}$; where ΔE is the variation achieved at each cooling step in the cost function E, defined as the distance between the expected assortative coefficient r and that quantified from the optimized network r_{net}: $E=|r_{\mathrm{net}}-r|$. The obtained randomized networks have thus an assortative coefficient very close to the expected value r ($E\leqslant 10^{-4}$ at the end of the cooling iterations), and precisely the same degree distribution $p(k)$ as the original network.

Although many cooling schemes have been discussed in the literature, in this work the temperature is simply multiplied by α at each cooling step. Cooling is done if either the number of accepted exchanges of edges since the last cooling exceeds N_{succ}, or the total number of configurations visited during the current cooling step exceeds N_{tot} [18]. As the temperature is decreased, the system configuration is annealed to the minimum of the cost function. For all the examples of our study, we have set $\alpha=0.9$, $N_{\mathrm{succ}}=5N$ and $N_{\mathrm{tot}}=N/2$, where N is the size of the network.

By monitoring the ratio λ_N/λ_2 of the coupling matrix G, we can now study the synchronizability of a class of networks with the same degree distributions but different assortative coefficients r. The used class of scale-free networks is obtained by a generalization of the preferential attachment growing procedure introduced in Ref. [20]. Namely, starting from $m+1$ all to all connected nodes, at each time step a new node is added with m links. These m links point to old nodes with probability $p_i=\frac{k_i+B}{\sum_j(k_j+B)}$, where k_i is the degree of the node i, and B is a tunable real parameter, representing the initial attractiveness of each node. This procedure allows a selection of the γ exponent of the power law scaling in the degree distribution [21] [$p(k)\sim k^{-\gamma(B,m)}$] with $\gamma(B,m)=3+\frac{B}{m}$ in the thermodynamic ($N\to\infty$) limit. While the average degree is by construction $\langle k\rangle=2m$ (thus independent on B), the heterogeneity of the degree distribution can be strongly modified by B.

For comparison, we also apply our analysis to a class of networks with a high homogeneous degree distribution, represented by random (RND) networks obtained via the rewiring procedure proposed in Ref. [22]. Here, starting from a ring lattice of N nodes connected with their k nearest neighbors, we substitute all nearest neighbor connections with links randomly pointing to other nodes in the network. For all the studied networks, the heterogeneity of weights is controlled by varying the parameter θ in the coupling term, whereas the r coefficient is tuned by applying the SA procedure described above.

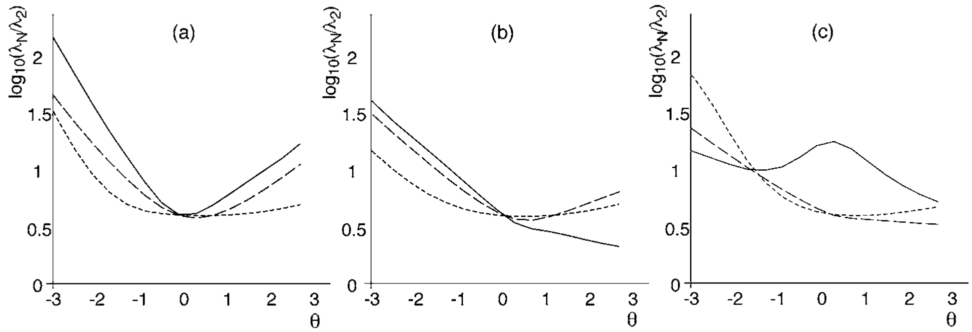

FIG. 1. λ_N/λ_2 (in logarithmic scale) vs the dimensionless parameter θ for (a) disassortative networks ($r=-0.3$), (b) topologies with $r=0$, and (c) assortative wirings ($r=0.3$). Solid and dashed lines correspond to the SF case with $B=0$ and $B=35$, respectively. Dotted lines indicate the results for RND networks. In all cases $m=5$, and the reported values refer to an average over 10 realizations of networks with $N=1000$ nodes. Random networks have $\langle k \rangle = 2m$, identical to that of SF networks.

Figure 1 shows the logarithm of λ_N/λ_2 as a function of the parameter θ for SF and RND networks with different degrees of assortative mixing. The first observation is that, the curves of λ_N/λ_2 for RND networks display a minimum at $\theta \simeq 0$, independently of the degree mixing. For disassortative SF topologies, a minimum is also observed at $\theta \simeq 0$. In contrast, for assortative ($r=0.3$) SF networks, the value of λ_N/λ_2 decreases with $\theta > 0$ with inflection points (computed numerically) located at $\theta \simeq -0.5$ and $\theta \simeq 1$ for $B=0$, and at $\theta \simeq 1$ for $B=35$. These results suggest that the differences between homogeneous and heterogeneous degree distributions for inducing an optimal synchronized behavior depend on the values of r. In all our results, N has been varied from 200 to 1000 without significant qualitative differences.

The effect of assortative mixing on synchronization is further illustrated in Fig. 2: Fig. 2(a) reports the behavior of $\ln(\lambda_N/\lambda_2)$ vs r for the case $\theta=0$ (i.e., when the links are weighted with the node degrees), indicating that a small negative degree mixing ($-0.3 \leqslant r \leqslant 0.2$) may provide better topologies for inducing a synchronized behavior than assortative wirings (thus fully recovering the results of Ref. [17]); while curves of Fig. 2(b) indicate that for other weighted coupling scheme (here $\theta=3$), the effect of degree-degree correlations strongly depends on the degree distribution of the networks. It is important to emphasize that, even though the coefficient r can take hypothetical values between $[-1,1]$, the obtained limits were often unable to span the whole range, basically due to the constraints imposed by the degree distribution $p(k)$ of the considered network. This explains the different limits of r for the topologies considered.

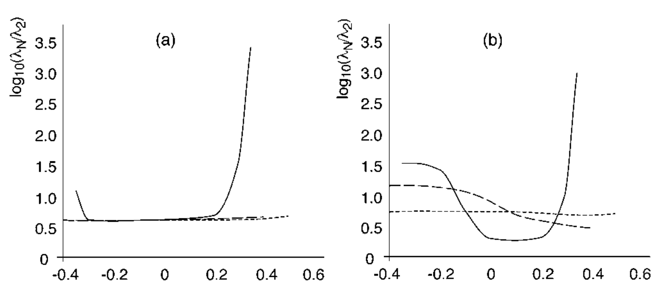

FIG. 2. λ_N/λ_2 (in logarithmic scale) vs r with a weighted coupling scheme given by (a) $\theta=0$ and (b) $\theta=3$. Same stipulations as in the caption of Fig. 1.

Recent works suggest that negative degree-degree correlations are an emerging property of networks when the propensity for inducing a synchronized behavior is optimized [17]. To further investigate the effect on synchronization of such disassortative topologies, we have tuned the value of the coefficient r for SF and RND networks towards its minimum value (theoretically $r \to -1$) and computed the ratio λ_N/λ_2 for different weighted coupling schemes (different values of θ). For this purpose, we have used the SA scheme described above, where the cost function was simply set as the assortative coefficient estimated from the network $E=r_{\text{net}}$. This randomization scheme allows reaching the minimum value of the degree-degree correlation of a network while its degree distribution remains unchanged.

Figure 3 reports the behavior of $\ln[\lambda_N(n)/\lambda_2(n)]$ vs $r(n)$ at each cooling step n of the SA procedure described in the paragraph above, for RND and SF networks and different values of θ. The crucial observation is that, as r reaches its lowest value, the ratio λ_N/λ_2 considerably increases for all values of θ. This is a remarkable result, insofar as it indicates that although a minimization of the ratio λ_N/λ_2 yields negative degree-degree correlations [17], a further minimization of the assortative coefficient induces a strong destabilization of the synchronous state.

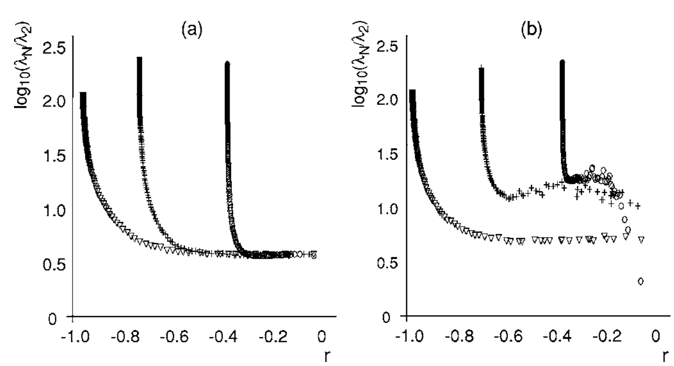

FIG. 3. Values of λ_N/λ_2 (in logarithmic scale) vs r obtained at each cooling step of the minimization procedure, for a weighted coupling scheme given by (a) $\theta=0$ and (b) $\theta=3$. In both graphs the curves with diamonds, crosses and down-pointing triangles refer to SF networks with $B=0$, $B=35$, and RND networks, respectively.

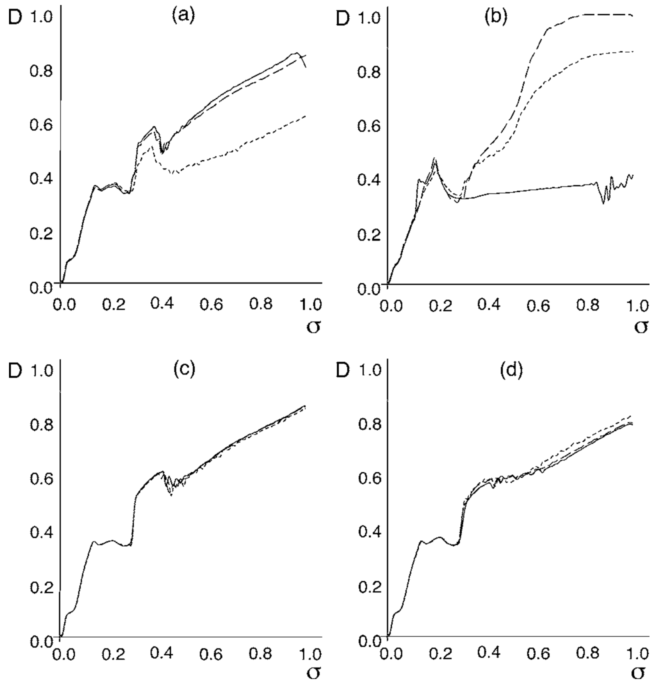

FIG. 4. The parameter D vs σ for networks of coupled chaotic quadratic maps. Solid, dashed, and dotted lines correspond to networks with $r=-0.3$, $r=0$, and $r=0.3$, respectively. The weighted coupling schemes considered are SF topologies $[B=0]$ with (a) $\theta=0$ and (b) $\theta=3$ and RND wirings with (c) $\theta=0$ and (d) $\theta=3$. Data refers to averages over 10 different realizations of networks with $N=1000$.

While our study focused on normalized coupling schemes, results for the non-normalized case $\mathcal{K}=1$ (not reported here) show the following changes in the scenario: for all the networks, the curves of λ_N/λ_2 have a pronounced minimum at $\theta\simeq-1$ whatever the value of r is. Furthermore, RND networks provide better topologies for inducing a synchronized behavior than SF networks, for all values of θ. Although the optimal conditions for synchronization can be obtained for disassortative wirings, a further minimization of parameter r also induce a strong destabilization of the synchronous state.

Finally, we illustrate our arguments by examples of different degree-correlated weighted networks of coupled maps. The dynamics is ruled by Eq. (1), with the local dynamics given by the quadratic map $\mathbf{f}(x)=1-ax^2$ set at the chaotic regime with $a=1.99$, and the coupling function $\mathbf{H}[\mathbf{x}]=\mathbf{f}(x)$. The appearance of the synchronous state can be monitored by looking at the time average (over a window of length L) of the parameter $D=1-\frac{1}{L}\sum_{t=1}^{L}(\varepsilon_t)^2$, where $(\varepsilon_t)^2=\sum_{i=1}^{N}[(x_t^i-\bar{x}_t)^2/\sum_{i=1}^{N}(x_t^i)^2]$ ($\bar{x}_t$ is the average amplitude at a given step time t). If all maps evolve independently, $D\sim0$. In contrast, if their motions are fully synchronized then $D=1$.

Figure 4 shows the behavior of D vs the coupling strength σ for SF and RND topologies, and for various values of r and θ. As expected, the observed synchronization scenario reflects qualitatively the results of Fig. 2. Indeed, when the SF networks are considered with $\theta=0$ [curves in Fig. 4(a)], the conditions $r\leqslant0$ provide a better synchronization behavior than the case $r>0$. Conversely, for $\theta=3$ [Fig. 4(b)] the synchronization scenarios for SF wirings with $r\geqslant0$ is clearly worse than topologies with $r<0$.

Synchronization properties of RND networks are indicated by curves in Figs. 4(c) and 4(d). These topologies are clearly less sensitive to degree-degree correlations than those of SF networks. The condition for $\theta=0$ [the ensemble of curves in Fig. 4(c)] provides, however, a better synchronization behavior than the case $\theta=3$ [curves in Fig. 4(d)].

Whereas for small σ, the SF and RND topologies do not show any appreciable qualitative difference, as the coupling strength becomes larger, uncorrelated ($r=0$) SF topologies provide larger values of D than RND wirings, independently of the weighting coupling scheme. In contrast, for networks with an assortative coefficient $r\neq0$, such differences are not trivial and they depend on the parameter θ.

In conclusion, we addressed the question of whether the propensity for synchronization of networks is affected by degree-degree correlations. The results show that the degree distribution, the weighted coupling scheme and the degree-degree correlations among the nodes compete in a nontrivial way to determine optimal synchronous behavior in weighted networks. In particular, we found that tuning the assortative coefficient of a network towards its minimum value induces a destabilization of the synchronous state.

Our approach may provide hints for the design of coupling matrices that warrant the stability of synchronized state in weighted networks. Results presented here may give insights into the mechanisms of real degree-correlated networks (as metabolic or epidemiological networks), that underlie the transmission and synchronization of information.

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